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Resonant periodic orbits and the semiclassical energy spectrum

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Abstract. The semiclassical density of states depends, according to the periodic-orbit sum formula, on the linear stability of the orbits. This means, however, that contributions from the marginally stable or 'resonant' orbits, which necessarily accompany stable ones, diverge unphysically. The remedy for a system of two degrees of freedom is found to lie in the classical non-linear normal forms for periodic orbits, which describe how satellite periodic orbits coalesce with the central one as resonance is approached ($\varepsilon \to 0$). Through these forms the resonant contributions are expressed as diffraction integrals (the first few being 'diffraction catastrophes') uniformly valid in ε and $\hbar$, and finite even for $\varepsilon \to 0$ provided $\hbar \neq 0$. An extension is proposed to incorporate, jointly, multiple resonances found in repetitions of orbits.

1. Introduction

Periodic orbits are trajectories which close and retrace themselves. For a typical Hamiltonian system with more than one degree of freedom (not an integrable one) such orbits are isolated: neighbouring orbits are not periodic. They may be stable or unstable depending on whether the neighbours stay close or escape away, and the stability is quantified in terms of the eigenvalues of a *stability matrix* $\mathbf{M}$ describing the linearisation about the periodic orbit in a *Poincaré section map* (see e.g. Henon 1983). The matrix $\mathbf{M}$ has unit determinant, so that for a system with two degrees of freedom its eigenvalues are $\exp(\pm i\alpha)$ for a stable orbit. This is the case we shall consider.

Simple (*first-order*) resonance occurs at $\alpha = 0$, so that both the eigenvalues are unity. Then $\mathbf{M}$ represents a simple shear mapping in the phase plane (it has one degenerate real eigenvector, representing a continuous line of fixed points through the origin in the linear approximation). Higher orders of resonance occur at all rational angles $\alpha = 2\pi n/m$ (n, m coprime) because, although $\mathbf{M}$ itself is not then resonant, its mth power $\mathbf{M}^m$, corresponding to the mth *repetition* of the periodic orbit, *is resonant.* For $m \geq 3$ the resonant matrix $\mathbf{M}^m$ is the identity (every point in the q, p plane is then fixed under $\mathbf{M}^m$) so that $\mathbf{M}$ itself represents (a linear distortion of) a rotation by $2\pi n/m$ in the q, p plane. For a second-order resonance, $m = 2$, however, the eigenvalues are still real (both -1), so that there is still a degenerate real eigenvector and $\mathbf{M}$ represents a simple shear with inversion.

The semiclassical theory of Gutzwiller (1971), in which the quantum density of states

$$n(E) = \sum_k \delta(E - E_k) \tag{1}$$

is given as a sum over the *isolated* periodic orbits of energy E and their repetitions, breaks down for resonant stable orbits. This is because the amplitude of an orbit in the sum is determined by the linearisation around it, which at resonance predicts a *continuum* of periodic orbits. The amplitude of a resonant orbit diverges. The remedy is to rederive the semiclassical approximation keeping the non-linear part of the Poincaré map which is not required off resonance. The effect is to replace the *continuum* by a *coalescence* of a finite set of 'satellite' periodic orbits onto the central one. The resulting amplitude is non-singular. In § 2 we rederive the periodic orbit sum and in § 3 describe the normal forms for each order of resonance. In § 4 we analyse the contributions of the resonances. It turns out that the contribution exactly at resonance can be identified where applicable with that obtained by Richens (1982) for an integrable system. The problem of large iterations, for which there may be several simultaneous near resonances, is dealt with in § 5.

2. The periodic-orbit sum

In this section the periodic-orbit sum formula is derived in full, avoiding the approximation for the amplitudes which breaks down for resonant orbits. The conclusion of the section is that the usual formula for these (16) is replaced by the integral (18).

Given the time-dependent Green function

$$K(\boldsymbol{q}, \boldsymbol{q}', t) = \sum_k \psi_k(\boldsymbol{q})\psi_k^*(\boldsymbol{q}') \exp(-\mathrm{i}\hbar^{-1}E_k t) \tag{2}$$

in terms of the eigenvalues E_k and eigenfunctions ψ_k of the Hamiltonian, we obtain the density of states as

$$n(E) = \int \mathrm{d}\boldsymbol{q} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi\hbar} K(\boldsymbol{q}, \boldsymbol{q}, t) \exp(\mathrm{i}\hbar^{-1}Et). \tag{3}$$

The semiclassical expression for the Green function (see e.g. Berry and Mount 1972) is

$$K(\boldsymbol{q}, \boldsymbol{q}', t) = (2\pi\mathrm{i}\hbar)^{-1} \sum_j \left| \det\left(\frac{\partial^2 \sigma_j}{\partial \boldsymbol{q}\, \partial \boldsymbol{q}'}\right) \right|^{1/2} \exp(\mathrm{i}\hbar^{-1}\sigma_j(\boldsymbol{q}, \boldsymbol{q}', t) - \mathrm{i}\mu_j\pi/2) \tag{4}$$

where

$$\sigma_j(\boldsymbol{q}, \boldsymbol{q}', t) = \int_0^t (\boldsymbol{p}_j \cdot \dot{\boldsymbol{q}}_j - H(\boldsymbol{p}_j, \boldsymbol{q}_j))\, \mathrm{d}t' \tag{5}$$

is the classical action along the jth classical trajectory that takes the time t to reach $\boldsymbol{q}'$ from $\boldsymbol{q}$. If the action is a minimum along the trajectory, then $\mu_j = 0$. This is the case for Hamiltonians of the form $H = \boldsymbol{p}^2/2m + V(\boldsymbol{q}, t)$ for sufficiently short time intervals. For longer intervals there may be isolated *focal points* along the orbit, where one or more of the eigenvalues of $\partial^2\sigma_j/\partial\boldsymbol{q}\,\partial\boldsymbol{q}'$ are infinite; μ_j is then the total number of such points along the orbit.

The singularities of the Green function at the focal points can be avoided by Maslov's method (Berry 1983). In this we consider the coordinate-momentum Green function

$$K'(\boldsymbol{q}, \boldsymbol{p}', t) = (2\pi\mathrm{i}\hbar)^{-1} \sum_j \left| \det\left(\frac{\partial^2 \sigma_j}{\partial \boldsymbol{q}\, \partial \boldsymbol{p}}\right) \right|^{1/2} \exp(\mathrm{i}\hbar^{-1}\sigma_j(\boldsymbol{q}, \boldsymbol{p}', t) - \mathrm{i}\mu_j'\pi/2) \tag{6}$$

where $\sigma_j(\boldsymbol{q}, \boldsymbol{p}', t)$ is the Legendre transform of the classical action along the jth path. Where the determinants of both (4) and (6) are non-singular we obtain one from the other by taking the Fourier transform by the method of stationary phase. Near a focal point, where the determinant of (4) is singular but (6) is not, the correct semiclassical approximation for $K(\boldsymbol{q}, \boldsymbol{q}', t)$ is the Fourier transform of (6) with no stationary phase approximation.

The time integral of (4) in (3) will pick out the classical trajectories of energy E, since

$$\frac{\partial \sigma_j}{\partial t}(\boldsymbol{q}, \boldsymbol{q}', t) = -H_j. \tag{7}$$

Generally for any given $\boldsymbol{q}$, there will be discrete initial momenta $\boldsymbol{p}_j(\boldsymbol{q}, E)$ in which the classical trajectory returns to $\boldsymbol{q}$ after some time, and the momentum on the return passage will be quite different from $\boldsymbol{p}_j(\boldsymbol{q})$. However, the relations

$$\partial\sigma/\partial\boldsymbol{q}' = -\boldsymbol{p}' \qquad \partial\sigma/\partial\boldsymbol{q} = \boldsymbol{p} \tag{8}$$

imply that the phase of the d$\boldsymbol{q}$ integral in (3) is stationary only if the return momentum equals the initial one. Thus semiclassically the density of states reduces to a sum over the periodic orbits of the system and a zero time contribution (Berry 1983)

$$\bar{n}(E) = (2\pi\hbar)^{-2}\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\,\delta(E - H(\boldsymbol{q}, \boldsymbol{p})). \tag{9}$$

The contribution of a periodic orbit of period τ and its repetitions $m\tau$, where m is a a positive or negative integer, are evaluated in the appendix. The contribution for the mth repetition is

$$(2\pi\mathrm{i}\hbar)^{-3/2}\tau A_m \exp(\mathrm{i}\hbar^{-1}m2\pi J - \mu_m\pi/2) \tag{10}$$

where A_m is the orbit amplitude

$$A_m = \int \mathrm{d}Q \left|\frac{\partial^2 S_m}{\partial Q\,\partial Q'}\right|^{1/2} \exp(\mathrm{i}\hbar^{-1}S_m(Q, Q)) \tag{11}$$

and

$$2\pi J = \oint \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q} \tag{12}$$

once around the periodic orbit and S_m (Q, Q') is the generating function for the mth iteration of the Poincaré map in the neighbourhood of the periodic orbit.

The origin is a fixed point of the Poincaré map. The linear part of the map around the origin is given by

$$\begin{pmatrix} Q' \\ P' \end{pmatrix} = \mathbf{M}\begin{pmatrix} Q \\ P \end{pmatrix}. \tag{13}$$

We can relate the quadratic part of the generating function S_m to $\mathbf{M}^m$ by

$$\frac{\mathrm{d}^2 S_m(Q, Q)}{\mathrm{d}Q^2} = \det(\mathbf{M}^m - \mathbf{1})\frac{\partial^2 S_m(Q, Q)}{\partial Q\,\partial Q'} \tag{14}$$

so that, if we evaluate (11) by stationary phase, the orbit contribution becomes

$$(\tau/2\pi\hbar\mathrm{i})[\det(\mathbf{M}^m - \mathbf{1})]^{-1/2}\exp(\mathrm{i}\hbar^{-1}m2\pi J - \mu_m\pi). \tag{15}$$

This reduces to the familiar formula

$$\frac{-\mathrm{i}\tau}{4\pi\hbar\sin(m\alpha/2)}\exp(\mathrm{i}\hbar^{-1}m2\pi J) \tag{16}$$

where the phases are accounted for by the sign of the denominator.

As we saw in § 1, $\det(\mathbf{M}^m-\mathbf{1})=0$ for a resonance, so that the Gutzwiller formula (15) for the mth contribution of the resonant orbit diverges. It is then necessary in (11) to integrate keeping the necessary non-quadratic terms of $S_m(Q, Q)$. A further problem is that the exact non-linear map is close to the identity, which cannot be generated by the action $S_m(Q, Q')$. We avoid it by starting from the coordinate-momentum Green function (6) instead of (4) and so obtain the amplitude in the form

$$A_m=(2\pi\hbar\mathrm{i})^{-1/2}\int \mathrm{d}P\,\mathrm{d}Q\left|\frac{\partial^2 S_m}{\partial P\,\partial Q}\right|^{1/2}\exp[-\mathrm{i}\hbar^{-1}(S_m(P,Q)-PQ)] \tag{17}$$

where $S_m(P, Q)$ is the Legendre transform of $S_m(Q, Q)$. It will also prove convenient to use polar canonical coordinates (I, ϕ) instead of cartesian coordinates P, Q. The amplitude then takes the form

$$A_m=(2\pi\hbar\mathrm{i})^{-1/2}\int \mathrm{d}I\,\mathrm{d}\phi\left|\frac{\partial^2 S_m}{\partial I\,\partial \phi}\right|^{1/2}\exp[-\mathrm{i}\hbar(S_m(I,\phi)-I\phi)]. \tag{18}$$

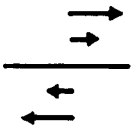
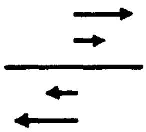
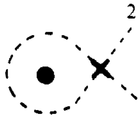
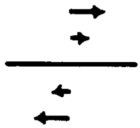
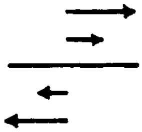
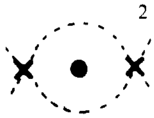
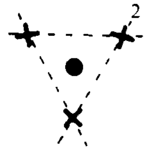
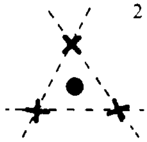
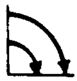
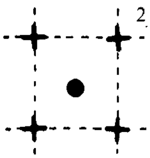
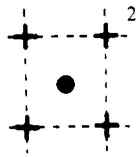
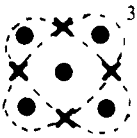
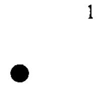
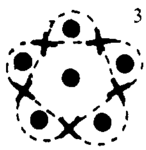
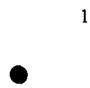
3. Normal forms for the resonant Poincaré maps

The phase function $S(P, Q')-PQ'$ has the beautiful property that it is stationary at each fixed point of the Poincaré map: $\partial S/\partial P=\partial S/\partial Q'=0 \Leftrightarrow Q'=Q,\ P'=P$. These correspond to stable periodic orbits in the case of extrema of the phase function and to ordinary unstable orbits for saddle points. (Unstable orbits with reflection, though, are extrema.) In particular the phase function is an extremum at the origin.

Off resonance the behaviour of the phase function near the origin is simply quadratic, but on resonance it is a flatter function of higher order. The task is to enumerate the generic local forms that it takes and perform the integrals to obtain the amplitudes. Only the local forms are required because we consider the semiclassical limit $\hbar\to 0$. The goal is a formula for the amplitude which is uniformly valid as $\hbar\to 0$ *and* $\varepsilon\to 0$ where ε measures deviation from resonance. The quadratic part of the function governs the linearisation and the generic way in which this breaks down, for a simple first-order resonance, is for the quadratic form to become degenerate, i.e. flat in one direction, with a consequent continuum of stationary points along it. This continuum is broken by the cubic variation of the generating function. Slightly off resonace the resulting function has an extremum at the origin and a saddle point nearby and as resonance is approached ($\varepsilon\to 0$) these coalesce (fold catastrophe) and annihilate to leave *no periodic orbit at all.* (This obliteration is not shared by higher-order resonance, below, where the central orbit always survives.)

Second and higher orders of resonance are not subject to the direct genericity argument, which yielded the simple coalescence above, because of the restriction that for mth order the mapping has an mth root (the ordinary one-traversal Poincaré map). Each order has just one (restrictedly generic) way in which it becomes degenerate, except fourth order which has a choice of two ways. They were classified by Meyer (1970) and are discussed by Arnold (1978, appendix 7). The contours and normal

Table 1. The mechanism of loss of stability of a periodic orbit with repetition number m, as described by its Poincaré map **M** and the corresponding normal form of its generating action.

Repetition number m	Linearised map **M** at resonance	Linearised map $\mathbf{M}^m$ at resonance	Fixed points of true $\mathbf{M}^m$ off resonance, $\varepsilon > 0$	Fixed points of true $\mathbf{M}^m$ off resonance, $\varepsilon < 0$	Normal form
1				0	$\varepsilon q^2 + q^3 + p^2$
2					$\varepsilon q^2 + q^4 + p^2$
3		Identity			$cq^2 + q^3 - qp^2$
4 ($\lvert c\rvert > \lvert a\rvert$)		Identity			$\varepsilon I + cI^2 + aI^2 \times \sin(4\phi)$
4 ($\lvert c\rvert < \lvert a\rvert$)		Identity			$\varepsilon I + cI^2 + aI^3 \times \sin(4\phi)$
5		Identity			$\varepsilon I + cI^2 + aI^{5/2} \times \sin(5\phi)$
>5	Natural extension of $m = 5$ case				

forms are indicated in table 1 both for $\varepsilon > 0$ and $\varepsilon < 0$. In all cases there is, as $\varepsilon \to 0$ from above, a coalescence of m satellite *saddle points*. For $m = 4$, $\lvert c\rvert < \lvert a\rvert$, and for $m > 4$, these are interspersed with m satellite extrema as well. When the satellite extrema are present, they annihilate the saddles at $\varepsilon = 0$, leaving only the central stable orbit for $\varepsilon < 0$. For $m = 3$ and for $m = 4$, $\lvert c\rvert > \lvert a\rvert$, the saddles simply pass straight through the origin as ε goes through zero. For $m = 2$ it is different again, the two saddles annihilate, but also convert the central extremum to a saddle point.

An important point about the relationship between the generating function picture and its consequent map picture is that the m satellite saddles correspond to a *single* unstable satellite periodic orbit which winds n times round the central one in m traversals. Likewise the satellite extrema are a *single* stable satellite orbit. Thus none of the diagrams in the table contains more than three periodic orbits and the actual number present, changing as ε goes through zero, is shown in the upper right corner of each diagram.

It is important to gain some further understanding of the origin of the normal forms for $m>3$. As stated earlier, in the linear approximation at resonance ($\varepsilon=0$) *all* orbits are periodic and can be labelled by their polar Poincaré section coordinates (I, ϕ). The function

$$m\tau H(I, \phi')=I\phi'-S(I, \phi') \tag{19}$$

has, at resonance, an interpretation: by standard perturbation theory it is the integral of a reduced Hamiltonian function (Arnold 1978, appendix 7) in phase space around that periodic orbit of the *linearised* system, which starts from and ends at (I, ϕ'). Off resonance ($\varepsilon \neq 0$) the linear part of the Poincaré map is no longer quite the identity, but a 'rotation' through a small angle ε which is generated by adding the term εI to $S(I, \phi')$. For small ε (19) still holds so that the phase function of the integrand of (18) is just the normal form of the averaged Hamiltonian $H(I, \phi)$ (Arnold 1978):

$$m\tau H(I, \phi')=\varepsilon I+\sum_{\nu=2}^{\nu<m/2} c_\nu I^\nu + aI^{m/2}\sin(m\phi'). \tag{20}$$

For $\varepsilon \neq 0$ we can neglect the ϕ-dependent term close to the origin, which is therefore surrounded by invariant constant-I circles, corresponding to thin tori in the full four-dimensional phase space. This is the *Birkhoff approximation or normal form.* If ε is small and $\varepsilon c_2<0$ there will be a torus with action

$$\bar{I}=-\varepsilon/2c_2+O(\varepsilon^2) \tag{21}$$

made up entirely of closed orbits. The basic effect of the ϕ-dependent term is to break this *quasitorus*, leaving only the isolated closed orbits already described. It may be noted that within the Birkhoff approximation the relation (19) between the Hamiltonian and the generating function for the Poincaré map is exact.

4. Resonant orbit amplitudes

It remains merely to insert the quoted normal forms into the exponent of (17) or (18). For $m=1, 2, 3$ the integral (17) yields standard diffraction catastrophe integrals (Berry and Upstill 1980), corresponding respectively to the fold, the cusp and the elliptic umbilic catastrophes (Poston and Stewart 1978). For $m=4$ the diffraction catastrophe integral is that of X_9, the first of the catastrophes involving a continuous variable parameter, the 'modulus' K. If the catastrophe potential function is written $x^4+y^4+Kx^2y^2$ then the two different cases in table 1 correspond to $K>-2$ and $K<-2$. For the higher resonances we insert (19) and (20) into (18) to obtain

$$A_m=(2\pi\hbar \mathrm{i})^{-1/2}\int_0^\infty \mathrm{d}I\int_0^{2\pi}\mathrm{d}\phi\,\exp\{\mathrm{i}\hbar^{-1}[\varepsilon I+\textstyle\sum c_\nu I^\nu+aI^{m/2}\sin(m\phi)]\}$$

$$=(-2\pi \mathrm{i}/\hbar)^{1/2}\int_0^\infty \mathrm{d}I\,J_0(\hbar^{-1}aI^{m/2})\exp[\mathrm{i}\hbar^{-1}(\varepsilon I+\textstyle\sum c_\nu I^\nu)] \tag{22}$$

where $J_0(x)$ is a Bessel function. Finally we scale the large parameter $\hbar^{-1}$ out of the Bessel function, so that with the change of variables

$$u = \hbar^{-2/m} I \tag{23}$$

the amplitude reduces to

$$A_m = (-2\pi i \hbar^{-1+4/m})^{1/2} \int_0^\infty du\, J_0(au^{m/2}) \times \exp\{i\hbar^{-1+4/m}[cu^2 + \varepsilon\hbar^{-2/m}u + O(\hbar^{2/m})]\} \tag{24}$$

where $c = c_2$ in (22) and we neglect higher terms in I.

If $m > 4$ the integrand has the form of a complex Gaussian with large frequency, modulated by a slowly varying function. The uniform asymptotic approximation for integrals of this form (Berry and Tabor 1976) leads to

$$A_m = (4\pi|c|i)^{-1/2} J_0(a\hbar^{-1}\bar{I}^{m/2}) \exp(-i\hbar^{-1}c\bar{I}^2) \int_{-\infty}^{-(|c|\hbar^{-1}/2)^{1/2}\bar{I}} \exp(i\beta x^2/2)\, dx + \varepsilon^{-1}(i\hbar/2\pi)^{1/2}[1 - J_0(a\hbar^{-1}\bar{I}^{m/2})] \tag{25}$$

where $\beta = \mathrm{sgn}(c)$ and $\bar{I}$ is given by (21) even when negative.

The amplitude takes on different forms in three different parameter regions.

(i) Far enough from resonance for $|a\bar{I}^{m/2}\hbar^{-1}| \gg 1$, the amplitude reduces to

$$A_m \simeq -i(4\pi c\hbar^{-1}a\bar{I}^{m/2})^{-1/2}\Theta(\bar{I}) \times \exp(-i\hbar^{-1}c\bar{I}^2)\cos(a\hbar^{-1}\bar{I}^{m/2} - \pi/4) + \varepsilon^{-1}(i\hbar/2\pi)^{1/2} \tag{26}$$

where $\Theta(x)$ is the unit step function. So for $I < 0$, (26) combined with (10) yields the Gutzwiller periodic orbit contribution (16), since

$$\varepsilon = m\alpha - 2n\pi. \tag{27}$$

For $\bar{I} > 0$ we also have the contributions

$$\begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{\tau}{2\pi} (2ca\hbar^{-1}\bar{I}^{m/2})^{-1/2} \exp[i\hbar^{-1}(m2\pi J - c\bar{I}^2 \pm a\bar{I}^{m/2})]. \tag{28}$$

Here the top ($-i$) contribution corresponds to the stable satellite orbit if $ca > 0$. The stability angle for this orbit is proportional to $\varepsilon^{m/4}$, in agreement with the classical result (Arnold 1978).

(ii) For large m there may be a region for which $|a\bar{I}^{m/2}\hbar^{-1}| \ll 1$, while

$$\hbar^{-1}c\bar{I}^2 = \hbar^{-1}\varepsilon^2/2c \gg 1 \tag{29}$$

and the amplitude is then

$$A_m \simeq -i(2c)^{-1/2}\Theta(\bar{I})\exp(-i\hbar^{-1}c\bar{I}^2/2) + \varepsilon^{-1}(i\hbar/2\pi)^{1/2}. \tag{30}$$

The central orbit still gives a non-resonant contribution (arising from the endpoint of the integral in (25)). But the satellites now yield a combined contribution, identical to that of the closed-orbit torus of the Birkhoff approximation. This family of closed orbits contributes with a higher power of $\hbar^{-1}$ than an isolated periodic orbit. In the Birkhoff approximation the Hamiltonian is integrable; this is the case treated by Berry and Tabor (1976, 1977a, b) who showed that the amplitude of a torus contribution is proportional to $|\det \partial^2 H/\partial \boldsymbol{I}^2|$. This determinant is diagonal in the Birkoff approximation and one of its elements is just $2c$.

(iii) As the resonance is approached we have $|\hbar^{-1}c\bar{I}^2/2| \ll 1$, so that (25) reduces to

$$A_m \simeq -\mathrm{i}(8c)^{-1/2} + (\hbar 8\pi)^{-1/2}\bar{I} \tag{31}$$

which can be identified with the resonant periodic orbit contribution obtained by Richens (1982) for an integrable system.

The three cases above can be summarised pictorially (figure 1) in terms of the 'Mexican hat' shape of the generating function for $m > 4$ and $\bar{I} > 0$. For case (i) both the area of the hat centre and the area of the ripples in the rim are large compared to $\hbar$. For case (ii) the area of the hat is large but the ripple area is not, and finally for case (iii) neither the hat area nor the ripple area are large.

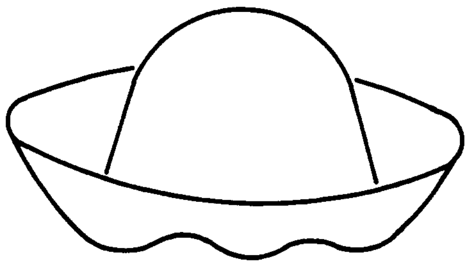


Figure 1. The generating action for the higher repetition numbers has the form of a 'Mexican hat' with ripples in the rim.

5. Many repetitions

Near resonances occur in the periodic-orbit sum in two ways. Resonances will appear as energy is varied along a one-parameter family of periodic orbits if we fix the repetition number m. Conversely, as m grows there will be near resonances, even though the energy and threfore the stability angle remain fixed. The resonance parameter given by (27) will be small for all m such that n/m is a good rational approximation of α.

So far we have restricted the analysis to the limit of small ε. But it can be argued as follows that the analysis is valid for sufficiently large ε to cope simultaneously with several near resonances. Recalling that according to (20), the constant c is proportional to m, we see that a near resonance arising from the violation of condition (29) happens even for finite ε as $m \to \infty$. In fact for sufficiently large m there will be more than one near-resonant quasitorus. The situation becomes clearer when analysed in terms of the coefficients of the resonant Hamiltonian (20),

$$\varepsilon' = \varepsilon/m\tau \qquad c' = c/m\tau \tag{32}$$

so that ε' is the difference between the frequency of rotation of the orbits surrounding the central orbit and a given rational frequency, in the linearised approximation. The quasitorus is given by $\bar{I} = -\varepsilon'/2c'$ with c' fixed. The condition for non-resonance then becomes

$$|m\tau H(\bar{I})| \gg \hbar. \tag{33}$$

Thus, for increasing m a given quasitorus will eventually contribute separately from the central orbit. However, for large m there will be more than one integer n such that the tori with frequencies $2\pi n/m\tau$ violate (29).

For a single near resonance the orbit contribution is correctly given by (18). However, the general form for angle variables, easily derived from the corresponding Green function (Berry and Tabor 1977a, b) is

$$A_m = (2\pi\hbar\mathrm{i})^{-1/2} \sum_n \int \mathrm{d}I\, \mathrm{d}\varphi \left| \frac{\partial^2 S_m}{\partial I\, \partial\varphi}(I, \varphi) \right|^{1/2} \exp[-\mathrm{i}\hbar^{-1}(S_m(I, \varphi) - I(\varphi + 2n\pi))]. \tag{34}$$

Inserting (20) and (19) in (34), with ε taken as the smallest value given by (27), we get stationary phases for many integrals. If m is not large, there will be at most one (with $n = 0$) for which the stationary points away from the origin cannot be treated as separate Gutzwiller orbits. For large m the converse is true: the contribution of the mth repetition of the periodic orbit will be simply the sum of several near-resonant contributions.

6. Conclusion

The semiclassical equivalence of the density of states with a sum of periodic orbits, derived by Gutzwiller more than a decade ago, remains the only general method available for the study of the spectrum of classically unintegrable systems. The small denominator problem for stable periodic orbits near resonance, marring the original formula, is correctly ascribed to an avoidable linearisation. Off resonance, no matter how close, the original formula is valid in the limit $\hbar \to 0$, but, since the distance ε from resonance is a continuous function of energy, there will be energies where the joint limit $\hbar \to 0$, $\varepsilon \to 0$ must be used, yielding amplitudes given by the normal forms of table 1.

If ε is taken to zero sufficiently faster than $\hbar$ then for $m > 4$ (and one of the two $m = 4$ cases) the satellite periodic orbits collapsing onto the central one contribute together as a quasitorus described by the first non-linear term in the Birkhoff approximation. Depending on ε and $\hbar$, it may or may not be possible to separate the amplitude of the quasitorus from that of the central orbit. If it can be separated, then it can be identified with that of a torus in the Berry–Tabor theory for integrable systems. It may thus be said that quantum mechanics has a smoothing effect on classical mechanics.

If the number of repetitions of the central orbit is sufficiently large, there will be many quasitori nearly resonant with it. The total amplitude for a given repetition is simply the sum of the amplitude of each near resonance. A given nearly resonant quasitorus for a repetition m will eventually contribute separately for some multiple repetition jm. However, by this time it will have been replaced by quasitori in its interior whose frequencies are better approximations of the frequency of the linearised system.

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Appendix

The evaluaton of the contribution of an individual periodic orbit to the density of states is simplified by a careful choice of coordinates, as follows. Generally periodic orbits cone in one-parameter (energy) families. We take P, Q as transverse canonical coordinates, $P = Q = 0$ being the orbit family itself, such that for P and Q and H constant we get closed curves. Their actions will be denoted by J and the angle coordinate along them by θ. This is specified by

$$\dot{\theta}_0 = \omega_0(J) = \frac{\partial}{\partial J} H(J, 0, 0). \tag{A1}$$

The Hamiltonian varies with P and Q, since $\partial H/\partial P = \dot{Q}$ and $\partial H/\partial Q = -\dot{P}$, which must be non-zero; otherwise the P and Q constant coordinate lines would be the neighbouring orbits, which are not generally closed. We can, however, choose the coordinate lines to be tangent to the orbits at $\theta = 0$. The return map $(P, Q)(0) \to (P, Q)(2\pi)$ is then the Poincaré map.

In the integral

$$-\mathrm{i}(2\pi\hbar)^{-2} \int \mathrm{d}Q\, \mathrm{d}\theta\, \mathrm{d}t \left| \det \frac{\partial^2 \sigma}{\partial \boldsymbol{q}\, \partial \boldsymbol{q}'} \right|^{1/2} \exp[\mathrm{i}\hbar^{-1}(\sigma + Et - \mathrm{i}\mu\pi/2)] \tag{A2}$$

the determinant is slowly varying. σ is a function of θ, θ', Q, Q' and t with

$$\partial\sigma/\partial\theta = J \qquad \partial\sigma/\partial\theta' = -J'. \tag{A3}$$

However, along the periodic orbit

$$\omega_0(J) = (\theta' - \theta)/t \tag{A4}$$

so the off-diagonal tori of the determinant are zero and

$$\det\left(-\frac{\partial^2\sigma}{\partial \boldsymbol{q}\, \partial \boldsymbol{q}'}\right) = \frac{1}{t} \frac{\partial J}{\partial \omega_0} \frac{\partial^2 \sigma}{\partial Q\, \partial Q'}. \tag{A5}$$

By the method of stationary phase

$$\int \mathrm{d}t \exp[\mathrm{i}\hbar^{-1}(\sigma + Et)] = \left(\frac{\mathrm{i}}{2\pi\hbar} \frac{\partial^2\sigma}{\partial t^2}\right)^{1/2} \exp[\mathrm{i}\hbar^{-1}(J(\theta' - \theta) + S(Q, Q'))] \tag{A6}$$

using (7) and

$$S(Q, Q') = \int_Q^{Q'} P\, \mathrm{d}Q. \tag{A7}$$

However,

$$\frac{\partial^2\sigma}{\partial t^2} = -\frac{\partial E}{\partial J} \frac{\partial J}{\partial t} = \omega_0 \frac{\partial J}{\partial \omega_0} \frac{\theta' - \theta}{t^2} = \frac{(2\pi)^2}{m\tau^3} \frac{\partial J}{\partial \omega_0} \tag{A8}$$

as $\theta - \theta' = m2\pi$ and $t = m\tau$. So (A2) becomes

$$[(2\pi)^5(\mathrm{i}\hbar)^3]^{-1/2}\tau \int \mathrm{d}Q\, \mathrm{d}\theta \left| \frac{\partial^2 S_m}{\partial Q\, \partial Q'} \right|^{1/2}$$
$$\times \exp[\mathrm{i}\hbar^{-1}(m2\pi J + S_m(Q, Q)) - \mathrm{i}\mu_m\pi/2]. \tag{A9}$$

The integrand is independent of θ, so the contribution of the mth repetition of a periodic orbit to the density of states is given by (10) and (11).

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